Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

of geometrical problems and not do as many students do, finish elementary geometry with the idea that it is a set of puzzles which are to be tried as a puzzle with little if any method.

These are briefly some indications of what I mean by principles and methods. A student thus equipped will have a set of tools and a knowledge of their use, so that he can carve out the solution of most any geometrical problem, whether old or new. Some of you have in your teaching been paying attention to these things and will agree with me that geometry thus studied and taught has not only great interest, but a value almost immeasurable.

SYRACUSE UNIVERSITY, SYRACUSE, N. Y.

ELEMENTARY LOGIC AS A BASIS FOR PLANE GEOMETRY.*

By EUGENE R. SMITH.

This paper is the report of an experiment which I have been trying in my classes, and it will on that account necessitate a frequent use of the personal pronoun.

Some nine years ago, my attention was attracted by the numerous mistakes in elementary logic which my pupils made. I began to wonder whether logic was a natural or an acquired habit, and to be on the lookout for errors of the same kind outside my classes.

I soon found that many men in everyday life were prone to use the converse or the obverse of a known statement, if it happened to suit their purposes, without the slightest qualm. Within a short period of time I heard two striking illustrations of this. In the first place a well-known speaker demonstrated that certain conditions were always followed by certain consequences, and ended up in a burst of eloquence with the statement that since those consequences were now present in our country, the conditions must be abroad in the land.

In the second case, the debating team of one of our large high schools attempted to win a debate by the very same kind of reasoning. As one of the judges I had to disallow the entire argument, and I could not but feel that in so doing, I was judging, not the boys of the debating team, but the school from which they came, for they had evidently been drilled within an inch of their lives along that very line of argument.

---

* Read at the annual meeting, Lancaster, Pa.

I then turned my attention to various mathematical texts, and I found that, while the sins of commission in this particular respect were not very numerous, the sins of omission were indeed frequent. One of the worst violators of the laws of logic gave the following brilliant example of how not to prove a theorem:

A transversal had been shown to be perpendicular to one of two lines, but not to the second. The consequent reasoning was: "If both lines had been perpendicular to this transversal they would have been parallel, but they are not both perpendicular to the transversal, so they are not parallel." There was no redeeming reference to the parallel axiom, nothing to relieve the atrocity of an assumed obverse.

I might add that my class investigations were not entirely along mathematical lines, but included such questions as this: Suppose John agrees to go to New York if Henry goes, and John is then seen in New York, is Henry necessarily there? Such questions aroused warm arguments between members of the classes, some boys seeing clearly the error of assuming the condition from the conclusion, while others failed at first to see any vital difference between a statement and the related statements.

After watching these various things for some time, I concluded that in so far as my experience showed, a correct logical sense was not the natural inheritance of the human race, but was rather the result of laborious groping after truth.

My next step was to try to improve the existing condition in my own classes, and I did this by a brief course in elementary logic at the start of plane geometry. This course was confined to the definitions of obverse, converse and contraposite in their simplest forms, and to the use of these relations in reasoning, more especially of course contraposite and converse reasoning.

After becoming accustomed to the use of the relations in simple concrete cases, the pupil was given the general case.

(1) If A, then B.

(2) If B, then A. (Converse.)

(3) If not *B*, then not *A*. (Contraposite.)

(4) If not *A*, then not *B*. (Obverse.)

The definitions of these relations are always given in as simple a form as possible, and the pupil is drilled until he can distinguish them with certainty. Such questions as " What is the converse of the observe of a statement " tests his ability to think the relations.

The next step is to establish the " law of contraposite," which is stated " The contraposite of any true statement is also true." This is done as follows:

Given the true statement " If *A*, then *B* " suppose " not *B* "; then there is the choice of " *A* " and " not *A*." But " *A* " is always followed by " *B*," which would give " If not *B*, then *A*, then *B* "; and this is a manifest contradiction. Therefore " not *A* " must be the correct conclusion, making the statement " If not *B*, then not *A*."

This proof may be required from the student or not, as a teacher wishes, but it should certainly be talked over with the class, for it is easily understood, and gives a finality to the law. This proof depends upon the fundamental law that a statement must not contradict itself, which is a special case of the law of excluded mean, but this is going too deep for use with a class.

The other important law is the law of converse, which is stated as follows:

" If statements whose conditions cover all possibilities, and no two of whose conclusions can be true at once, are known to be true, their converses are also true."

PROOF.

Given the true statements

If *A*, then *X*,
If *B*, then *Y*,
If *C*, then *Z*,
If *D*, then *W*,
etc.,

where *A*, *B*, *C*, *D*, etc., cover all possibilities, and no two of *X*, *Y*, *Z*, *W*, etc., can be true at once.

Suppose *X*, then neither *Y*, *Z*, *W* nor any other conclusion can be true, for no two conclusions can be true at once. Then by the law of contraposite no one of the conditions *B, C, D,* etc., can be true, and since the conditions cover all possibilities, *A,* the only remaining possibility must be true.

These two facts—that the contraposite of a true statement is also true, and that, while the converse of one statement can never be assumed, the converses of statements which fulfill the converse law are true—are drilled into the student by numerous examples, and are referred to in recitation often enough to keep them well in mind.

For the student who persists in assuming a converse, the following argument is usually effective:

" A donkey has a head, you have a head, therefore what must we conclude? "

The special case of the converse law, which involves greater, equal, and less, is discussed in full, and the pupil is taught that the obverse is true only when the converse is, for it is the contraposite of the converse.

It might be said that this study involves much less use of time than would be thought at first, and the time so used is more than made up before the end of plane geometry.

One of my most awful experiences occurred the first time I started teaching this to a class. I wished to use such simple logic that the text-books on the subject were of little use to me, and I was forced to make my own outline.

After having talked over the related statements with my class and having told them that the contraposite of a true statement was always true, I asked the class to make up familiar conditional statements, write their obverses, converses and contraposites, and verify our conclusions by experiment. The next day when I brought up the subject of contraposite, the brightest boy in the class raised his hand, and when called upon, told me that he had explained the lesson to his father the night before, and that his father had become so interested that they had looked up the subject in the Century Dictionary, only to find that the dictionary said that the contraposite was not always true, quoting an example to show this.

That was a heavy blow to my budding scheme, and I was forced to tell the class that I must examine what the Century said on the subject before I dared say it was wrong. I reassured them as best I might, but such an authority loomed large to all of us and the recitation was not a great success.

I hurried to the Century after school and read the passage, which was exactly as the boy had said. It did not take long, however, to see that the Century was indeed wrong, for not only was it possible to prove that the contraposite of any true statement was also true, but the example given, if looked into carefully, only confirmed this fact. The example was as follows:

" If the ink blots, I will not spill it."

Contraposite—" If I spill the ink, it will not blot."

I will spare you a discussion of the fallacy involved, but I was forced to convince that class that the Century was wrong, and when I had done it, I considered that pure logic had won a great victory. I need hardly say that a logical error in such an authority convinced me all the more that there was supreme need of some logical training.

When a class has learned to distinguish statements which are obverse, converse and contraposite, and has had impressed upon it the fact that true statements always go in contraposite pairs and that the obverse or the converse of a single statement is never safe to assume—although the obverses and converses of exclusive conditional statements covering all possibilities are true—the class is not only quite largely safeguarded against a frequent kind of error, but it is provided with a powerful weapon of attack. When I say that it is safeguarded against errors I do not take the position that mistakes of just this kind will not creep in—for they will. If there is any way to get perfect reasoning from pupils, I have not yet found it. I do mean that such errors are greatly decreased and that the ones which do occur are from carelessness, not ignorance, and are quickly recognized if the correctness of the step is questioned. A boy may add 2 and 3 and write 6—he knows better, and if the teacher questions the answer, he sees his error. So, here, if a boy assumes a wrong logical step, the question " What step in reasoning did you use?" quickly sets right the difficulty. The difference between pupils trained in this method and those wholly or even partly ignorant has been shown many times in my classes, when a pupil has entered the class after the logic had been finished. Such a pupil invariably used incorrect logic in discovering some of the new theorems, while the class as a whole distinguished clearly between the correct and incorrect logical steps. I might explain here that my classes work out the proofs of the theorems without a text-book, so weakness in logic shows more quickly than by the other method.

As a weapon of attack, contraposite stands alone. I believe it to be, next to the general method of classification and elimination, the most important single method in geometry. Instead of wasting time in teaching it as an extra method, the teacher saves time, for its applications are almost continuous throughout the subject, and, among other things, it does away entirely with the need of "reductio ad absurdam," which is purely contraposite reasoning without the merit of being recognized as such.

I think I am safe in saying that a pupil needs just three methods of attack for plane geometry: the classification and elimination method, the analysis method—used mainly for construction work—and the method of pure logic, which includes contraposite, converse and obverse reasoning.

The contraposite, of course, requires no method—so called—for the only proof needed is to state a known theorem and say that the required statement is its contraposite and must therefore be true.

Converses and obverses of single statements have a method of their own. If one is asked to prove the converse or the obverse of a single known statement, such a proof can be possible only if the single statement is exclusive. I can best show my meaning by an example:

A line through the midpoint of one side of a triangle parallel to a second side bisects the third side.

The converse form: A line through the midpoints of two sides of a triangle is parallel to the third side.

Now this converse is true, not alone because the original statement is, but because the line obtained in the first statement is the only one through those two points; that is, there is an exclusive element which can be added to the first statement. This then is the key to the converse or obverse method when applied to a single statement:

Draw the figure according to the conditions of the original statement, then draw it according to the conditions of the required statement; finally show that the two are identical for converse, are different for obverse. This will be more clearly shown by other illustrations a little later.

The use of converse reasoning when the theorems already proved cover all possibilities is too well known to need discussion. I might say, however, that the proof usually given is not complete, for the converses can be assumed only after the statements are shown to be exclusive statements covering all possibilities, and this is usually neglected.

I will illustrate the use of these methods in proving theorems, by the use of a few of the common theorems.

The subject of parallels is an excellent example of the use of each kind of logical reasoning. The pupil can very readily see that there is an interdependence between parallels and the angles formed with a transversal, and the investigation of this relation will cover the whole subject of parallels. I always treat the angles in sets of four rather than in alternate and exterior interior pairs, thus combining all the theorems on the equality of the angles formed into four theorems. The sets of four have been shown in a preliminary theorem to depend on an alternate interior pair.

Note.—These are stated in abbreviated form.

1. If the lines are not parallel, the angles are not equal.
2. If the angles are unequal, the lines are not parallel.
   This is the converse of (1).
3. If the angles are equal, the lines are parallel.
   This is the contraposite of (1).
4. If the lines are parallel, the angles are equal.
   This is the obverse of (1).

Very early in the geometry I have taken up the theorem that an exterior angle of a triangle is greater than either interior angle not adjacent to it, and (1) is evidently this same theorem. It is naturally the first to be taken up, for it involves—not the new idea of parallels—but the already familiar idea of intersecting lines. The pupil sees at once that when two lines are crossed by a transversal, a triangle is formed if the lines meet, and the inequality of the two alternate interior angles follows.

But the other three statements are all related to this one, and (3) is true at once because it is contraposite to (1).

There remain to prove only (2) and (4). These can be proved separately by the converse and obverse methods, or one can be proved, and the other will follow because they are contraposites. I usually prove (2) as the obverse of (3) and use contraposite reasoning to obtain (4). If this is done, the entire subject of parallels is covered by the exterior angle theorem, which is of value for other purposes, and one obverse proof, the other statements being contraposites.

The obverse method applied to (2) would be as follows:

If the angles are equal, the lines are parallel (3).

If a line is drawn through the same point on the transversal, making the angles unequal, it cannot be parallel, for there can be but one parallel to a line through a point.

To prove (4) by the converse method, the following proof could be used:

Given two parallels crossed by a transversal; if a line is drawn through the point where one of the parallels crosses the transversal, making equal angles with the transversal, that line will be parallel to the other of the two given parallels (3), and therefore will coincide with the given line through the same point, by the parallel axiom.

(2) and (4) are excellent examples of the method to be applied in proving the converse and the obverse of a single statement, and these proofs are probably both well worth talking over with a class, even though the use of contraposite makes but one of them necessary.

Two very simple examples of contraposites depend upon axioms.

"Two straight lines can intersect in but one point" is the contraposite of "Through two points there can be but one straight line."

"Lines parallel to the same line are parallel to each other" is proved by the contraposite of "Through a point there can be but one parallel to a given line."

As good an example of the use of the law of converse as occurs in the geometry is in the discussion of the interdependence of the sides and the angles of a triangle. Suppose that the two theorems, "If two sides of a triangle are equal, the opposite angles are equal," and "If two sides of a triangle are unequal, the opposite angles are unequal, the greater angle being opposite the longer side," have been proved; these facts can be arranged in this form, using a general triangle ABC:

If $AB > BC, \angle C > \angle A$;
If $AB = BC, \angle C = \angle A$;
If $AB < BC, \angle C < \angle A$.

These statements evidently cover all possibilities ($>, =, <$), and no two of the conclusions can be true at once; therefore their converses are true, and

$$\text{If } \angle C > \angle A, AB > BC;$$
$$\text{If } \angle C = \angle A, AB = BC;$$
$$\text{If } \angle C < \angle A, AB < BC;$$

which proves the two theorems in which the condition refers to the angles, the conclusion to the sides.

Whatever kind of reasoning a pupil is using he should first state definitely the known statements from which he is reasoning, and then should draw his conclusions, stating what kind of reasoning is being used. In using the law of converse, he should say that the conditions cover all possibilities, and that no two of the conclusions can be true at once before drawing the conclusions.

In conclusion I may say that the experiment has been a great success. I not only find the reasoning of my classes more logical—in fact more critically logical—but the classes discover new facts with much greater facility than before.

The records of pupils who have mastered these first principles of logic have been excellent, both in college entrance examinations, and in more advanced work taken up by many of them, and I feel that I have greatly strengthened my teaching of geometry by my experiment in adding elementary logic to the course taught.

Polytechnic Preparatory School,
Brooklyn, N. Y.

MATHEMATICAL HISTORICAL MATERIAL FROM THE EAST.*

By Bertha Lillian Broomell.

It is impossible to give in a few words any adequate idea of the rare and valuable mathematical material collected last year in the Far East by Professor David Eugene Smith while on his sabbatical leave. We can merely mention some of his most important acquisitions.

---

* Abstract of paper read at the annual meeting, Lancaster, Pa.